Rapid Communication

The bulk modulus and Poisson's ratio of "incompressible" materials

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Abstract

Poisson's ratio, v, is a fundamental parameter characterizing the mechanical behavior of a material. Because the ratio of the bulk to the shear modulus, B/G, becomes infinite when $v = 1/2$, it is often assumed that the bulk modulus becomes very large as a material approaches "incompressibility." This is incorrect; experimental results for viscoelastic materials show that changes in the bulk modulus are actually negligible as v approaches $1/2$. An analysis is performed to clarify the apparent conflict between the classic elastic equations and the experiments. At $v = 1/2$, the bulk modulus is shown to exhibit a singularity, but this is irrelevant to real materials.
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Poisson's ratio, v, defined as the ratio of the lateral contraction to the elongation, is of particular interest for materials near the glass transition, since v may exhibit a time or frequency dependence. Mathematically, when $v = 1/2$ the ratio of the bulk to shear modulus B/G is infinite and the system is described as incompressible. Experimentally, all materials are compressible (i.e., B is finite), and while the Poisson's ratio can approach $1/2$, it never actually equals $1/2$. In viscoelastic materials, when v is close to $1/2$, $G \ll B$ and the material is commonly called "incompressible". Interestingly, as $v \to 1/2$ the value of the B decreases: investigations of polymers in the softening zone of their viscoelastic spectrum show this behavior without exception; e.g., polymethyl methacrylate [1–3], polystyrene [2], polycarbonate [4], styrene-butadiene rubber [5], polyvinyl acetate [6], vulcanized natural rubber [7], nitrile-butadiene rubber [8], polyetherimide [9], an aromatic polyimide [10], and a polyurethane [11,12].

Fig. 1 shows representative data of the bulk modulus, shear modulus, and Poisson's ratio for polystyrene and polymethyl methacrylate [1]. The bulk and shear modulus were determined from longitudinal and shear wave speeds at 1 MHz, and Poisson's ratio was calculated from these data. This is the usual method of determining v, although there are other means, such as direct measurement of the lateral contraction accompanying uniaxial straining [2]. Conducting wave speed measurements through the softening zone presents special difficulties so most of the data are confined to glasses. The figure shows that as the polymers

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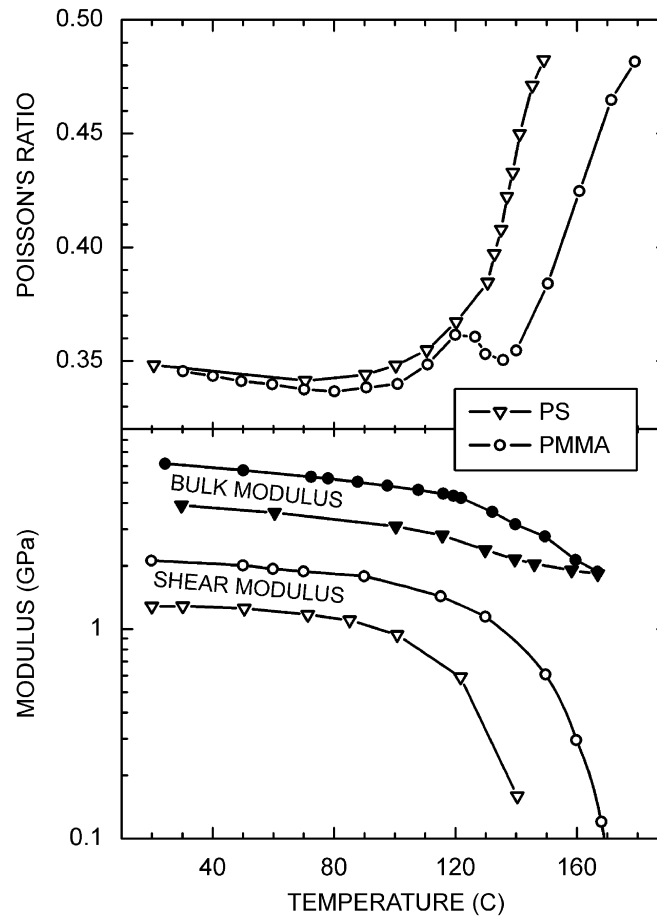
Fig. 1. Bulk and shear modulus, and Poisson's ratio, for polystyrene and polymethyl methacrylate as a function of temperature, determined from ultrasonic measurements at 1 MHz. Data from Ref. [1].

are heated, the bulk modulus drops by a factor of 2 or 3, the shear modulus decreases by orders of magnitude approaching zero, and Poisson's ratio approaches 1/2. This behavior is typical of viscoelastic materials. A recent review describes measurement techniques and discusses the shortcomings of the available experimental data [13].

Physically, as the temperature is increased through the glass transition, the distance between the molecules is expected to increase. The resulting decrease in the bulk modulus may be attributed to the consequent decrease in the slope of the interatomic potential. This increase in volume also increases the molecular mobility, decreasing the resistance to shear. This leads to a paradox: as a glass is heated and softens, it becomes more compressible as its Poisson's ratio becomes closer to the value associated with "incompressibility".

Mathematically, because B/G becomes infinite when $v = 1/2$, it is sometimes anticipated that the bulk modulus becomes anomalously large as a material approaches "incompressibility." For example, in a recent article in this journal expressions were developed for the complex Poisson's ratio and comparisons made to experimental data for polymers [12]. The relevant equation (Eq. (21) in Ref. [12]) was justified for *nearly incompressible* materials "because $B_d \rightarrow \infty$ as $v_d \rightarrow 1/2$" (subscript d refers to storage components of the dynamic elastic constants). This is at odds with the experiments, cited above, showing that B_d *decreases* as $v_d \rightarrow 1/2$. We demonstrate mathematically herein that for v close to 1/2, changes in the bulk modulus are small, in agreement with the observations of real materials.

The behavior of the elastic constants as Poisson's ratio approaches one-half can be deduced using the classic equations of elasticity. The ratio of the shear to the bulk modulus is [14]

$$\frac{G}{B} = \frac{3(1 - 2v)}{2(1 + v)}. \tag{1}$$

Hence $G/B = 0$ when $v = 1/2$. Differentiation gives

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{G}{B}\right) = \frac{1}{B^2}\left[B\frac{\mathrm{d}G}{\mathrm{d}v} - G\frac{\mathrm{d}B}{\mathrm{d}v}\right] = \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{3(1-2v)}{2(1+v)}\right) = -\frac{9}{2(1+v)^2}, \tag{2}$$

Eq. (2) indicates that the slope of G/B approaches -2 as $v \to 1/2$. This result is obtained from the fundamental relationships and involves no assumptions; thus, its validity is general. Rearranging Eq. (2) gives

$$\frac{\mathrm{d}G}{\mathrm{d}v} - \frac{G}{B}\frac{\mathrm{d}B}{\mathrm{d}v} = -\frac{9B}{2(1+v)^2}. \tag{3}$$

Fig. 1 shows the shear modulus of a viscoelastic material decreasing dramatically as Poisson's ratio approaches $1/2$. Therefore, using the assumption that $G \to 0$ as $v \to 1/2$ gives

$$\lim_{v \to 1/2}\left(\frac{\mathrm{d}G}{\mathrm{d}v}\right) = -2B, \tag{4}$$

Eq. (3) can be evaluated in the limit as $v \to 1/2$ to find

$$\lim_{v \to 1/2}\left(\frac{\mathrm{d}B}{\mathrm{d}v}\right) = \lim_{v \to 1/2}\left[\frac{B}{G}\frac{\mathrm{d}G}{\mathrm{d}v} + \frac{9B^2}{2G(1+v)^2}\right] = \lim_{v \to 1/2}\left[\frac{B}{G}(-2B) + \frac{9B^2}{2G(1+v)^2}\right] = \left[\frac{-2B^2}{G} + \frac{2B^2}{G}\right], \tag{5}$$

where we avoided the indeterminate form via substitution of Eq. (4) in the penultimate term of Eq. (5); the result is

$$\lim_{v \to 1/2}\left(\frac{\mathrm{d}B}{\mathrm{d}v}\right) = 0. \tag{6}$$

The important results are Eqs. (2), (4), and (6), but it must be remembered that while Eq. (2) is general, Eqs. (4) and (6) assume $G \to 0$ as $v \to 1/2$.

Such a situation ($G \to 0$ as $v \to 1/2$) occurs, for example, in a simple liquid that cannot support a shear stress at long times. However, for entangled polymers the dynamic G is not zero, although the shear modulus is three orders of magnitude lower than its value in the glassy state. Under these circumstances Eq. (2) holds; however, Eqs. (4) and (6) are only approximations, whose accuracy depends on the material under consideration.

The forgoing analysis clarifies the apparent conflict between classical elasticity and experiment. For a viscoelastic material above the glass transition, $G \neq 0$, and it follows from Eq. (1) that when $v = 1/2$ and $G/B = 0$, then $B = \infty$. The common assumption follows that the bulk modulus becomes large as $v \to 1/2$, which is contrary to experimental results. However, Eq. (6) shows that this assumption is not true. In fact the opposite is to be expected—B is invariant as this limit is approached from below, as demonstrated in Fig. 1, which shows that the bulk modulus is nearly constant as v approaches $1/2$. The contradiction is resolved by recognizing that the bulk modulus must exhibit a singularity at $v = 1/2$, a hypothetical condition not applicable to any actual material. Thus for a viscoelastic material the bulk modulus will change negligibly as incompressibility is approached. At $v = 1/2$ the function jumps discontinuously to infinity, but since all materials are compressible, the singularity is of no practical import. This interpretation is consistent with experiments cited above.

The derivation presented herein will be compared in detail to experimental results in a future publication.

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